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# Four-loop moments of the heavy quark vacuum polarization function in perturbative QCD

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**Abstract.** New results at four-loop order in perturbative QCD for the first two Taylor coefficients of the heavy quark vacuum polarization function are presented. They can be used to perform a precise determination of the charm- and bottom-quark mass. Implications for the value of the quark masses are briefly discussed.

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## 1 Introduction

Two-point correlators are of central importance for many theoretical and phenomenological investigations in quantum field theory. As a consequence they are studied in great detail in the framework of perturbative calculations. Due to simple kinematics (only one external momentum) even multi-loop calculations can be performed. The results for all physically interesting diagonal and non-diagonal correlators including *full* quark mass dependence are available up to $\mathcal{O}\left(\alpha_s^2\right)$ [1–3].

At four-loop order the two-point correlators can be considered in two limits: in the high energy limit massless propagators need to be calculated, and in the low energy limit vacuum integrals ("tadpole diagrams") arise. The evaluation of the latter in three-loop approximation has been pioneered in [4] and automated in [5].

Recently first results for physical quantities, which are related to four-loop tadpole diagrams, have been obtained. The four-loop matching condition for the strong coupling constant $\alpha_s$ at a heavy quark threshold has been calculated in [6,7]. The four-loop QCD contribution to the electroweak $\rho$-parameter induced by the singlet diagrams of the $Z$-boson self-energy has been computed in [8].

The detailed knowledge of the heavy quark correlator is important for the precise determination of heavy quark masses with the help of QCD sum rules. As known from [9,10], the determination of the charm- and bottom-quark mass is further improved, if the four-loop corrections, hence $\mathcal{O}(\alpha_s^3)$, for the lowest Taylor coefficients of the vacuum polarization function are available. The subset of four-loop contributions to the lowest two moments, which involve two internal loops from massive and massless fermions coupled to gluons, hence of $\mathcal{O}(\alpha_s^3\, n_f^2)$, has already been calculated in [11]. The symbol $n_f$ denotes the number of active quark-flavors, contributing through fermion loops inserted into gluon lines. The terms being proportional to $\alpha_s^j\, n_l^{j-1}$ are even known to all orders $j$ in perturbative QCD [12]. The symbol $n_l$ denotes the number of light quarks, considered as massless. In this paper the complete four-loop contributions originating from non-singlet diagrams for the first two Taylor coefficients are presented. Singlet contributions have been studied in [13–15].

In general, the tadpole diagrams encountered in these calculations contain both massive and massless lines. As is well known, the computation of the four-loop $\beta$-function can be reduced to the evaluation of four-loop tadpoles composed of completely massive propagators only. Calculations for this case have been performed in [16–18].

The outline of this paper is as follows. In Sect. 2 we briefly introduce the notation and discuss generalities. In Sect. 3 we discuss the reduction to master integrals, describe the solution of the linear system of equations, give the result for the lowest two moments and discuss briefly the impact on the quark mass determination. Our conclusions and a brief summary are given in Sect. 4.

## 2 Notation and generalities

The correlator $\Pi^{\mu\nu}(q)$ of two currents is defined as

$$\Pi^{\mu\nu}(q,j) = \mathrm{i}\int \mathrm{d}x e^{\mathrm{i}qx}\langle 0|Tj^{\mu}(x)j^{\nu}(0)|0\rangle\,, \qquad (1)$$

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with the current $j^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x)$ being composed out of the heavy quark fields $\Psi(x)$. The function $\Pi^{\mu\nu}(q)$ is conveniently written in the form

$$\Pi^{\mu\nu}(q) = \left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right) \Pi(q^2)\,. \tag{2}$$

The vanishing of the longitudinal contribution, as well as the confirmation of $\Pi^\mu_\mu(q^2=0)=0$, has been used as a check of the calculation. The function $\Pi(q^2)$ is of phenomenological interest, because it can be related to the ratio $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ with the help of dispersion relations:

$$\Pi(q^2) = \frac{1}{12\pi^2}\int_0^\infty \mathrm{d}s \frac{R(s)}{(s-q^2)} \ \text{mod subtr.} \tag{3}$$

Performing the derivative of (3) with respect to $q^2$ one obtains on the one hand "experimental" moments:

$$\mathcal{M}_n^{\text{exp}} = \int \mathrm{d}s \frac{R(s)}{s^{n+1}}\,, \tag{4}$$

which can be evaluated from the $R$-ratio. On the other hand one can define "theoretical" moments:

$$\mathcal{M}_n^{\text{th}} = Q_q^2 \frac{9}{4}\left(\frac{1}{4\overline{m}_q^2}\right)^n \overline{C}_n\,, \tag{5}$$

which are related to the Taylor coefficients $\overline{C}_n$ of the vacuum polarization function:

$$\overline{\Pi}(q^2) = \frac{3Q_q^2}{16\pi^2}\sum_{n\geq 0}\overline{C}_n \overline{z}^n\,, \tag{6}$$

with $\overline{z} = q^2/(4\overline{m}^2)$. Symbols carrying a bar denote that renormalization has been performed in the $\overline{\text{MS}}$ scheme. The Taylor expansion in $q^2$ around $q^2=0$ leads to massive tadpole integrals. The first and higher derivatives can be used for a precise determination of the charm- and bottom-quark mass. However, also the lowest expansion coefficient $\overline{C}_0$ has an interesting physical meaning, since it relates the coupling of the electromagnetic interaction in different renormalization schemes.

Sum rules as a tool for the determination of the charm- and bottom-quark mass have been suggested since long in [19]. This method has then been applied later also to the determination of the bottom-quark mass [20]. One of the most precise determinations of the charm- and bottom-quark mass being based on sum rules in connection with the calculation of three-loop moments in perturbative QCD has been performed in [9], with the values $\overline{m}_c(\overline{m}_c) = 1.304(27)$ GeV and $\overline{m}_b(\overline{m}_b) = 4.191(51)$ GeV as results for the charm- and the bottom-quark mass.

It is convenient to define the expansion of the Taylor coefficients $\overline{C}_n$ of the vacuum polarization function in the strong coupling constant $\alpha_s$ as

$$\overline{C}_n = \overline{C}_n^{(0)} + \left(\frac{\alpha_s}{\pi}\right)^1 \overline{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \overline{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \overline{C}_n^{(3)} + \ldots\,. \tag{7}$$

Due to the distinct mass hierarchy of the quarks in the standard model, one can consider for the above moments one species of quarks as massive ($n_\text{h}=1$) and all lighter ones as massless. The number of active quarks $n_f$ is then decomposed according to $n_f = n_\text{l} + n_\text{h}$. For convenience the symbol $n_\text{h}$ is kept explicitly in the following.

## 3 Calculations and results

The Feynman diagrams have been generated with the help of the program QGRAF [21]. After performing the expansion in the external momentum $q$ all integrals can be expressed in terms of 55 independent vacuum topologies with additional increased powers of propagators and additional irreducible scalar products.

In order to reduce this host of integrals to a small set of master integrals the traditional integration-by-parts (IBP) method has been used in combination with Laporta's algorithm [22, 23]. The resulting system of linear equations has been solved with a FORM3 [24–26] based program, in which partially also ideas described in [23, 27, 28] have been implemented. The rational functions in the space-time dimension $d$, which arise in this procedure, have been simplified with the program FERMAT [29]. Masking of large integral coefficients has been implemented, a strategy also adopted in the program AIR [30]. In order to achieve the reduction to 13 master integrals more than 31 million IBP equations have been generated and solved. This leads to integral-tables with solutions for around five million integrals. Furthermore all symmetries of the 55 independent topologies have been taken into account in an automated way, by reshuffling the powers of the propagators of a given topology in a unique way. Taking into account symmetries is important in order to keep the size of the integral-tables under control.

The first four master integrals shown in Fig. 1 can be calculated completely analytically in terms of $\Gamma$-functions. The fifth integral ($T_{52}$) can be obtained from the results of [4, 31].

The remaining eight master integrals (Fig. 2) have been determined with high precision numerics with the difference equation method [23] in [32]. Independently they have

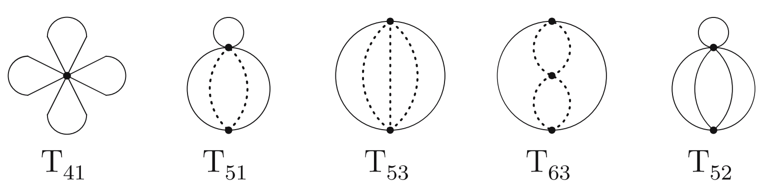


**Fig. 1.** Factorized or analytically known master integrals. The *solid* (*dashed*) *lines* denote massive (massless) propagators

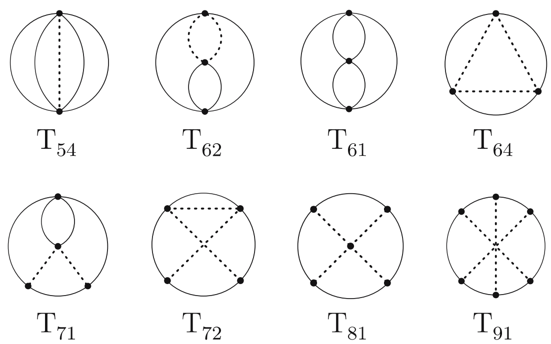


**Fig. 2.** Master integrals where only a few terms of their $\varepsilon$-expansion are known analytically. The *solid* (*dashed*) *lines* denote massive (massless) propagators

been determined in [33] by constructing an $\varepsilon$-finite basis. Some of these master integrals have also been calculated in [8, 11, 34, 35].

Inserting the master integrals and performing renormalization of the strong coupling constant $\alpha_s$, the external current and the mass $m = \overline{m}(\mu)$ leads to the following result ($\mu = \overline{m}$):

$$\begin{aligned}
\overline{C}_0^{(3)} = {} & n_l n_h \Big( -\frac{2}{9} a_4 + \frac{7043}{34992} - \frac{1}{108}\log^4(2) + \frac{\pi^2}{108}\log^2(2) + \frac{49}{12960}\pi^4 - \frac{127}{324}\zeta_3 \Big) \\
& + n_h^2 \left( \frac{610843}{2449440} - \frac{661}{2835}\zeta_3 \right) + n_l^2 \left( \frac{17897}{69984} - \frac{31}{162}\zeta_3 \right) \\
& + n_l \Big( -\frac{50}{81} a_4 - \frac{71629}{46656} - \frac{25}{972}\log^4(2) + \frac{25}{972}\log^2(2)\pi^2 + \frac{8533}{116640}\pi^4 - \frac{21343}{3888}\zeta_3 \Big) \\
& + n_h \Big( -\frac{28364}{405} a_4 - \frac{83433703}{8164800} - \frac{7091}{2430}\log^4(2) + \frac{7091}{2430}\log^2(2)\pi^2 + \frac{14873}{18225}\pi^4 - \frac{14509529}{340200}\zeta_3 + \frac{5}{3}\zeta_5 \Big) \\
& + \frac{64}{3} a_5 + \frac{12007}{243} a_4 - \frac{8572423579}{604661760} + \frac{12007}{5832}\log^4(2) - \frac{8}{45}\log^5(2) - \frac{12007}{5832}\log^2(2)\pi^2 + \frac{8}{27}\log^3(2)\pi^2 \\
& - \frac{1074967}{699840}\pi^4 + \frac{34}{135}\log 2\pi^4 + \frac{\pi^6}{486} + \frac{53452189}{349920}\zeta_3 - \frac{28}{243}\zeta_3^2 - \frac{37651}{648}\zeta_5 - \frac{1}{432}T_{54,3} + \frac{7}{1296}T_{62,2} \, ,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\overline{C}_1^{(3)} = {} & n_l n_h \Big( -\frac{116}{243} a_4 + \frac{262877}{787320} - \frac{29}{1458}\log^4(2) + \frac{29}{1458}\log^2(2)\pi^2 + \frac{1421}{174960}\pi^4 - \frac{38909}{58320}\zeta_3 \Big) \\
& + n_h^2 \left( \frac{163868}{295245} - \frac{3287}{7290}\zeta_3 \right) + n_l^2 \left( \frac{42173}{98415} - \frac{112}{405}\zeta_3 \right) \\
& + n_h \Big( -\frac{1394804}{8505} a_4 - \frac{27670774337}{1414551600} - \frac{348701}{51030}\log^4(2) + \frac{348701}{51030}\log^2(2)\pi^2 + \frac{1447057}{765450}\pi^4 - \frac{95617883401}{943034400}\zeta_3 + \frac{128}{27}\zeta_5 \Big) \\
& + n_l \Big( -\frac{4793}{7290} a_4 - \frac{9338899}{2099520} - \frac{4793}{174960}\log^4(2) + \frac{4793}{174960}\log^2(2)\pi^2 + \frac{372689}{839808}\pi^4 - \frac{48350497}{1399680}\zeta_3 \Big) \\
& - \frac{127168}{1215} a_5 - \frac{22152385}{61236} a_4 + \frac{237787820456749}{380936908800} - \frac{22152385}{1469664}\log^4(2) + \frac{15896}{18225}\log^5(2) \\
& + \frac{22152385}{1469664}\log^2(2)\pi^2 - \frac{15896}{10935}\log^3(2)\pi^2 - \frac{29962031}{176359680}\pi^4 - \frac{67558}{54675}\log 2\pi^4 + \frac{60701}{1071630}\pi^6 \\
& + \frac{282830677079}{881798400}\zeta_3 - \frac{242804}{76545}\zeta_3^2 - \frac{653339}{2430}\zeta_5 - \frac{5849}{272160}T_{54,3} + \frac{60701}{408240}T_{62,2} \, ,
\end{aligned} \tag{9}$$

where Riemann's zeta-function $\zeta_n$ and the polylogarithm function $\mathrm{Li}_n(1/2)$ are defined by

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad \text{and} \quad a_n = \mathrm{Li}_n(1/2) = \sum_{k=1}^{\infty} \frac{1}{2^k k^n} \, . \tag{10}$$

The two numerical constants that appear in (8) and (9) were obtained in (19) and (20) of [33]:

$$\begin{aligned}
T_{54,3} &= -8445.8046390310298\ldots \quad \text{and} \\
T_{62,2} &= -4553.4004372195263\ldots \, .
\end{aligned} \tag{11}$$

Numerically the coefficients $\overline{C}_0$ and $\overline{C}_1$ are given by

$$\begin{aligned}
\overline{C}_0 = {} & \left(\frac{\alpha_s}{\pi}\right) 1.4444 \\
& + \left(\frac{\alpha_s}{\pi}\right)^2 (1.5863 + 0.1387 n_h + 0.3714 n_l) \\
& + \left(\frac{\alpha_s}{\pi}\right)^3 (0.0252 n_h n_l + 0.0257 n_l^2 - 0.0309 n_h^2 \\
& - 3.3426 n_h - 1.2112 n_l + 1.4186) + \ldots
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\overline{C}_1 = {} & 1.0667 + \left(\frac{\alpha_s}{\pi}\right) 2.5547 \\
& + \left(\frac{\alpha_s}{\pi}\right)^2 (0.2461 + 0.2637 n_h + 0.6623 n_l) \\
& + \left(\frac{\alpha_s}{\pi}\right)^3 (0.1658 n_h n_l + 0.0961 n_l^2 + 0.0130 n_h^2 \\
& - 6.4188 n_h - 2.9605 n_l + 8.2846) + \ldots
\end{aligned} \tag{13}$$

Using the relation (5) for the first moment at three- and four-loop approximations one can assess the influence of the new four-loop order on the values of the charm- and bottom-quark mass. We first summarize the current status of the charm- and bottom-quark masses as obtained in [9] from the first moment evaluated to order $\alpha_s^2$:

$$\overline{m}_c(3\,\mathrm{GeV}) = 1.027 \pm 0.002\mathrm{GeV}\,, \tag{14}$$

$$\overline{m}_b(10\,\mathrm{GeV}) = 3.665 \pm 0.005\mathrm{GeV}\,. \tag{15}$$

Note that here and below we display only the uncertainties coming from the variation of the renormalization scale $\mu$ in the region $\mu = 10 \pm 5$GeV for the bottom-quark and $\mu = 3 \pm 1$GeV for the charm-quark respectively. The experimental error is larger and is discussed in [9].

Let us discuss the influence of the newly computed correction on the charm- and bottom-quark masses. In our analysis we will closely follow [9]. In particular, we will borrow from that work the value of the first "experimental" moment as defined in (4). The latest experimental information on $R(s)$ which appeared after publication of [9] will be taken into account in a future study.

The inclusion of the four-loop contribution to the function $\overline{C}_1$ leads for the case of the charm-quark to the following modification of (14):

$$\overline{m}_c(3\,\mathrm{GeV}) = 1.023 \pm 0.0005\mathrm{GeV}\,. \tag{16}$$

For the case of the bottom-quark our result reads

$$\overline{m}_b(10\,\mathrm{GeV}) = 3.665 \pm 0.001\mathrm{GeV}\,. \tag{17}$$

Thus, the four-loop correction does not change the value of $\overline{m}_b(10\,\mathrm{GeV})$ at all (within our accuracy) but does lead to significant decrease of the theoretical uncertainty.

## 4 Summary and conclusion

The calculation of Taylor expansion coefficients of the vacuum polarization function is important for a precise determination of the charm- and bottom-quark mass. In this work we have presented a new result at four-loop order in perturbative QCD for the first two coefficients of the Taylor expansion. For the computation the traditional IBP method in combination with Laporta's algorithm has been used in order to reduce all appearing integrals onto a small set of master integrals.

With the knowledge of the four-loop contributions the theoretical uncertainty is well under control in the view of the current and foreseeable precision of experimental data.

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**Note added in proof:**
The results of our calculations as expressed in (9) have been confirmed in the work [36]. where also a strong reduction of the theoretical error due to the unphysical scale dependence has been found.